A NOTE ON REGULARITY OF SOLUTIONS TO DEGENERATE ELLIPTIC EQUATIONS OF CAFFARELLI-KOHN-NIRENBERG TYPE

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Abstract. We establish Hölder continuity of weak solutions to degenerate critical elliptic equations of Caffarelli-Kohn-Nirenberg type.

1. Introduction

Our purpose is to establish Hölder continuity of weak solutions to

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = \frac{f}{|x|^{bp}}, \text{ in } \Omega \subset \mathbb{R}^N, \tag{1.1}$$

where Ω is an open, $N \geq 3$ and a, b, and p satisfy

$$\begin{aligned} -\infty < a < \frac{N-2}{2}, &\quad a \leq b \leq a+1 \\ p = p(a,b) = \frac{2N}{N-2(1+a-b)}&. \end{aligned} \tag{1.2}$$

We denote by $\mathcal{D}_a^{1,2}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_a^{1,2}(\Omega)} := \left(\int_\Omega |\nabla u|^2 |x|^{-2a}\right)^{1/2}.$$

For a given weight ω we denote by $L^p(\Omega,\omega)$ the space of functions u such that

$$\|u\|^p_{L^p(\Omega,\omega)} := \int_\Omega |u|^p \omega(x) < \infty.$$

The space $H_a^1(\Omega)$ is defined to be the closure of $C^\infty(\bar{\Omega})$ with respect to

$$\|u\|^2_{H_a^1(\Omega)} := \int_\Omega |x|^{-2a}\left(|\nabla u|^2 + |u|^2\right).$$

Our interest in these problems arose because of their relation to nonlinear, degenerate elliptic equations stemming from the family of Caffarelli-Kohn-Nirenberg inequalities [2]: if a, b, and p satisfy (1.2) then we have for all $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$

$$\left(\int |u|^p |x|^{-bp}\right)^{1/p} \leq \mathcal{C}_{a,b,N}\left(\int |\nabla u|^2 |x|^{-2a}\right)^{1/2}. \tag{1.3}$$

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For best constants and existence of minimizers in (1.3) we refer to [3]. Due to its characterization any minimizer $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$, if it exists, is a weak solution to

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = \frac{K(x)|u|^{p-2}u}{|x|^{bp}} \text{ in } \mathbb{R}^N, \tag{1.4}$$

where $K(x) \equiv$ const for some appropriate constant. The exponent $p = p(a, b, N)$ is the critical exponent in (1.3) and shares many features with the critical Sobolev exponent, e.g. (1.4) possesses for $K(x) \equiv K(0)$ a dilation symmetry, which gives rise to a noncompact manifold of weak radial solutions for $K(0) > 0$. In order to study problem (1.4) for non-constant functions K using for instance a degree argument, Hölder estimates for weak solutions of (1.1) are an important tool (see [7]).
Regularity properties of weak solution to degenerate elliptic problems with more general weighted operators of the form $\mathrm{div}\,(\omega(x)\nabla(\cdot))$ are studied in [5, 6, 8] (see also the references mentioned there). The classes of weights ω treated there include the class (QC) of weights

$$\omega(x) = |\mathrm{det}T'|^{1-2/N},$$

where $T : \mathbb{R}^N \to \mathbb{R}^N$ is quasi-conformal (see [6, 8] for a definition). In fact our weights $|\cdot|^{-2a}$ are associated with quasi-conformal transformations $T_a(x) := x|x|^{-2a/(N-2)}$. The right-hand sides studied in [5, 6, 8] are either zero or in divergence form, e.g. Hölder continuity of weak solutions to

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = \mathrm{div}\,(F) \text{ in } \Omega$$

is established in [5] assuming $|F||x|^{2a} \in L^p(\Omega, |x|^{-2a})$ for some $p > \max(N - 2a, N, 2)$. We derive Hölder estimates for weak solutions to (1.1) in terms of f, because a sharp relation of the integrability of f and its representation in divergence form F in the various weighted spaces is not obvious. We compare weak solutions of (1.1) with μ_a-harmonic functions, which are by definition weak solutions of

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = 0 \text{ in } \Omega,$$

where Hölder regularity is known (see for instance [8]) and prove

Theorem 1.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in H_a^1(\Omega)$ weakly solves (1.1), that is*

$$\int_\Omega |x|^{-2a}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega |x|^{-bp} f\varphi\, dx \quad \forall\, \varphi \in H_{0,a}^1(\Omega).$$

Assume a, b and p satisfy (1.2), $b < a+1$, and $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$. Then $u \in C^{0,\alpha}$ for any $\alpha \in (0,1)$ satisfying

$$\alpha < \min(\alpha_h, 1) \text{ and } \alpha < \begin{cases} \left(\frac{N-2}{2} - a\right)\left(p - 2 - \frac{p}{s}\right) & \text{if } b \geq 0 \\ \frac{N}{p}\left(p - 2 - \frac{p}{s}\right) & \text{if } b < 0 \end{cases},$$

where α_h is the regularity of μ_a-harmonic functions given in Theorem 2.1 below. Moreover, for any $\Omega' \Subset \Omega$ there is a constant $C = C(N, a, \alpha, \Omega, dist(\Omega', \Omega), s)$ such that

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C\left\{\|u\|_{L^2(\Omega,\, d\mu_a)} + \|f\|_{L^s(\Omega, |x|^{-bp})}\right\}$$

For the nonlinear problem (1.4) we use a De Giorgi-Moser type iteration procedure as in [1] and obtain

Theorem 1.2. *Let a, b and p satisfy (1.2) and $u \in D_a^{1,2}(\mathbb{R}^N)$ be a weak solution to*

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)\frac{|u|^{p-2}u}{|x|^{bp}}, \quad x \in \Omega \tag{1.5}$$

where $K \in L^\infty(\Omega)$. Then $u \in L^s_{\mathrm{loc}}(\Omega, |x|^{-bp})$ for any $s \in [p, +\infty[$. Moreover, u is Hölder continuous with Hölder exponent given in Theorem 1.1.

Remark 1.3. *While completing this note we learned that in [4] weighted q-Laplacian equations of the form*

$$-\operatorname{div}(|x|^{-qa}|\nabla u|^{q-2}\nabla u) = g$$

are studied. Under assumption (1.2) Hölder regularity of weak solutions to equation

$$-\operatorname{div}(|x|^{-qa}\nabla u) = \frac{f}{|x|^{bp}}, \text{ in } \Omega \subset \mathbb{R}^N,$$

is shown if $a = b$, $a > -1$, and $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$. Theorem 1.1 extends this result to the full range for a and b in the case $q = 2$.

2. Preliminaries

We collect some properties of the weighted measure $\mu_a := |x|^{-2a}\,dx$ and μ_a-harmonic functions. We refer to [6, 8] for the proofs.

- The measure μ_a satisfies the doubling property, i.e. for every $\tau \in (0,1)$ there exists a constant $C_{(2.1)}(\tau)$ such that

$$\mu_a(B(x,r)) \le C_{(2.1)}(\tau)\mu_a(B(x,\tau r)) \tag{2.1}$$

- A Poincaré-type inequality holds, i.e. there is a positive constant $C_{(2.2)}$ such that any $u \in D_a^{1,2}(\mathbb{R}^N)$ satisfies

$$⨍_{B_r(x)} |u - u_{x,r}|^2 \, d\mu_a \le C_{(2.2)}\, r^2 ⨍_{B_r(x)} |\nabla u|^2 \, d\mu_a, \tag{2.2}$$

  where $u_{x,r}$ denotes the weighted mean-value

$$u_{x,r} := ⨍_{B_r(x)} u \, d\mu_a = \frac{1}{\mu_a(B_r(x))}\int_{B_r(x)} u(x)\, d\mu_a.$$

Concerning μ_a- harmonic functions we have the following results.

Theorem 2.1 (Thm. 3.34 in [8](p. 65), Thm. 6.6 in [8](p. 111))**.**
There are constants $C_{(2.3)}(N,a)$ and $\alpha_h = \alpha_h(N,a) \in (0,1)$ such that if u is μ_a-harmonic in $B_r(x_0) \subset \mathbb{R}^N$ and $0 < \rho < r$ then

$$\textit{ess-sup}_{B(x_0,\frac{r}{2})}|u| \le C_{(2.3)} ⨍_{B(x_0,r)} |u|^2 \, d\mu_a, \tag{2.3}$$

$$osc(u, B_\rho(x_0)) \le 2^{\alpha_h}\big(\frac{\rho}{r}\big)^\alpha osc(u, B_r(x_0)). \tag{2.4}$$

Consequently, μ_a-harmonic functions are Hölder continuous.

We will call a function $u \in D^{1,2}_{a,loc}(\mathbb{R}^N)$ weakly super μ_a-harmonic in Ω, if for all nonnegative $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega |x|^{-2a}\nabla u \nabla\varphi \ge 0. \tag{2.5}$$

Theorem 2.2 (Thm 3.51 in [8](p. 70))**.** *There exist positive constants $s = s(N,a)$ and $C_{(2.6)} = C_{(2.6)}(N,a)$ such that if u is nonnegative and weakly super μ_a-harmonic in Ω and $B_{2r}(x_0) \subset \Omega$ we have*

$$\text{ess inf}_{B_{\frac{r}{2}}(x_0)} u \geq C_{(2.6)} \Big(\fint_{B_r(x_0)} u^s \, d\mu_a \Big)^{\frac{1}{s}}. \tag{2.6}$$

We use the two theorems above to derive

Lemma 2.3. *For any ball $B_r(x_0)$ there is a constant $C_{(2.7)}(B_r(x_0))$ such that any μ_a-harmonic function u in $B_r(x_0)$ satisfies*

$$\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \leq C_{(2.7)} \Big(\frac{\rho}{r}\Big)^{2\alpha_h - 2} \fint_{B_r(x_0)} |\nabla u|^2 \, d\mu_a, \tag{2.7}$$

where $\alpha_h \in (0,1)$ is given in Theorem 2.1.

Proof. To prove the claim we may assume that $0 < \rho < (1/4)r$ and that u has mean-value zero in $B_r(x_0)$. We take a cut-off function $\xi \in C_c^\infty(B_{2\rho}(x_0))$ such that $\xi \equiv 1$ in $B_\rho(x_0)$, $0 \leq \xi \leq 1$, $\|\nabla \xi\|_\infty \leq 2\rho^{-1}$ and define $\varphi := \xi^2(u - u(x_0))$. Testing with φ and using Hölder's inequality we get

$$\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \leq \int_{B_r(x_0)} |\nabla \xi|^2 (u - u(x_0))^2 \, d\mu_a \leq \|u - u(x_0)\|^2_{\infty, B_{2\rho}(x_0)} \mu_a\big(B_{2\rho}(x_0)\big)\rho^{-2}. \tag{2.8}$$

From (2.8) and (2.4) we infer

$$\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \leq C \Big(\frac{\rho}{r}\Big)^{2\alpha} osc(u, B_{\frac{r}{2}}(x_0))^2 \rho^{-2} \leq C \Big(\frac{\rho}{r}\Big)^{2\alpha} \rho^{-2} \fint_{B_r(x_0)} |u|^2 \, d\mu_a$$

Finally, since u has mean-value zero in $B_r(x_0)$ the Poincaré inequality (2.2) yields the claim. □

3. Growth of local integrals

We give a weighted version of the Campanato-Morrey characterization of Hölder continuous functions.

Theorem 3.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in L^2(\Omega, d\mu_a)$ satisfies*

$$\fint_{B_r(x)} |u(y) - u_{x,r}|^2 \, d\mu_a \leq M^2 r^{2\alpha} \quad \textit{for any } B_r(x) \subset \Omega \tag{3.1}$$

and some $\alpha \in (0,1)$. Then $u \in C^{0,\alpha}(\Omega)$ and for any $\Omega' \Subset \Omega$ there holds

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \Big\{ M + \|u\|_{L^2(\Omega, |x|^{-2a})} \Big\}$$

where $C = C(N, a, \alpha, \Omega, dist(\Omega', \Omega))$ is a positive constant independent of u.

Proof. Denote $R_0 = \text{dist}(\Omega', \partial\Omega)$. Using the triangle inequality and integrating in $B_{r_1}(x_0)$ we have for any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \leq R_0$

$$\begin{aligned} &\big|u_{x_0,r_1} - u_{x_0,r_2}\big|^2 \\ &\quad \leq \frac{2}{\mu_a(B_{r_1}(x_0))} \Big\{ \int_{B_{r_1}(x_0)} \big|u(x) - u_{x_0,r_1}\big|^2 \, d\mu_a + \int_{B_{r_2}(x_0)} \big|u(x) - u_{x_0,r_2}\big|^2 \, d\mu_a \Big\}. \end{aligned}$$

Using assumption (3.1) we obtain

$$\left|u_{x_0,r_1} - u_{x_0,r_2}\right|^2 \le \frac{2M^2}{\mu_a(B_{r_1}(x_0))}\left\{\mu_a(B_{r_1}(x_0))r_1^{2\alpha} + \mu_a(B_{r_2}(x_0))r_2^{2\alpha}\right\}. \tag{3.2}$$

For any $R \le R_0$ we take $r_1 = 2^{-(i+1)}R$ and $r_2 = 2^{-i}R$ in (3.2). The doubling property (2.1) then gives

$$\left|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}\right| \le 2M^2\Big(1 + C_{(2.1)}(N,a)2^{2\alpha}\Big)2^{-2(i+1)\alpha}R^{2\alpha}.$$

We sum up and get for $h < k$

$$\left|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}\right| \le \frac{C(N,a,\alpha)M}{2^{h\alpha}}R^{\alpha}. \tag{3.3}$$

The above estimates prove that $\{u_{x_0,2^{-i}R}\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$, hence it converges to some limit, denoted as $\hat{u}(x_0)$. The value of $\hat{u}(x_0)$ is independent of R, which may be seen by analogous estimates. Consequently, from (3.3) we have that

$$\left|u_{x_0,r} - \hat{u}(x_0)\right| \le C(N,a,\alpha)Mr^{\alpha} \quad \forall\, x_0 \in \Omega'. \tag{3.4}$$

By the Lebesgue theorem we infer

$$u_{x,r} = \frac{|B_r(x)|}{\int_{B_r(x)}|y|^{-2a}\,dy}\cdot\frac{\int_{B_r(x)}|y|^{-2a}u(y)\,dy}{\frac{1}{|B_r(x)|}} \underset{r\to 0^+}{\longrightarrow} u(x), \quad \text{a. e. in } \Omega'.$$

Hence $\hat{u} = u$ a. e. in Ω' and (3.4) gives

$$\left|u_{x_0,r} - u(x_0)\right| \le C(N,a,\alpha)Mr^{\alpha} \quad \forall\, x_0 \in \Omega', \tag{3.5}$$

which implies that $u_{x,r}$ converges to u uniformly in Ω'. Since $x \mapsto u_{x,r}$ is a continuous function, we conclude that u is continuous in Ω'. From (3.5) we have

$$|u(x)| \le C(N,a,\alpha)MR^{\alpha} + |u_{x,R}| \quad \forall\, x \in \Omega',\ \forall\, R \le R_0.$$

Thus u is bounded in Ω' with the estimate

$$\sup_{\Omega'}|u| \le c(N,a,\alpha,\Omega,\operatorname{dist}(\Omega',\Omega))\Big\{M + \|u\|_{L^2(\Omega,|x|^{-2a})}\Big\}. \tag{3.6}$$

Let us now prove that u is Hölder continuous. Let $x, y \in \Omega'$ with $|x-y| = R < \frac{R_0}{2}$. Assume that $|x| < |y|$. Then we have

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms are estimated by (3.5), whereas for the last term we have

$$|u_{x,2R} - u_{y,2R}|^2 \le 2\big\{|u_{x,2R} - u(\xi)|^2 + |u(\xi) - u_{y,2R}|^2\big\}$$

and integrating with respect to ξ over $B_{2R}(x) \cap B_{2R}(y) \supseteq B_R(x)$ we obtain

$$|u_{x,2R} - u_{y,2R}|^2 \le \frac{2}{\mu_a(B_R(x))}\Big(M^2\mu_a(B_{2R}(x))2^{2\alpha}R^{2\alpha} + M^2\mu_a(B_{2R}(y))2^{2\alpha}R^{2\alpha}\Big).$$

Since x is closer to 0 than y, we have that $\mu_a(B_{2R}(y)) \le \mu_a(B_{2R}(x))$ and hence

$$|u(x) - u(y)| \le C(N,a,\alpha)M|x-y|^{\alpha}.$$

If $|x-y| > \frac{R_0}{2}$ we can use estimate (3.6) thus finding

$$|u(x) - u(y)| \le 2\sup_{\Omega'}|u| \le c2^{\alpha}\left[M + \frac{1}{R_0^{\alpha}}\|u\|_{L^2(\Omega,|x|^{-2a})}\right]|x-y|^{\alpha}.$$

The proof is thereby complete. □

Corollary 3.2. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in H^1_a(\Omega)$ satisfies*

$$\fint_{B_r(x)} |\nabla u|^2 \, d\mu_a \le M^2 r^{2\alpha-2} \quad \text{for any } B_r(x) \subset \Omega$$

and some $\alpha \in (0,1)$. Then $u \in C^{0,\alpha}(\Omega)$ and for any $\Omega' \Subset \Omega$ there holds

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le c\Big\{M + \|u\|_{L^2(\Omega,|x|^{-2a})}\Big\}$$

where $c = c(N, a, \alpha, \Omega, dist(\Omega', \Omega)) > 0$.

Proof. The proof follows from Theorem 3.1 and the Poincaré type inequality in (2.2). □

Proof of Theorem 1.1. Let $w \in u + H^1_{0,a}(B_r(x_0))$ be the unique solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a}\nabla w) = 0 & \text{in } B_r(x_0) \\ w\big|_{\partial B_r(x_0)} = u. \end{cases} \tag{3.7}$$

Clearly the function $v = u - w \in H^1_{0,a}(B_r(x_0))$ weakly solves

$$-\mathrm{div}\,(|x|^{-2a}\nabla v) = \frac{f}{|x|^{bp}} \quad \text{in } B_r(x_0).$$

Testing the above equation with v and using Hölder's inequality and (1.3), we get

$$\int_{B_r(x_0)} |\nabla v|^2 \, d\mu_a x \le \mathcal{C}_{a,b,N}\bigg(\int_{B_r(x_0)} |x|^{-bp}|f|^{\frac{p}{p-1}}\bigg)^{\frac{p-1}{p}}\bigg(\int_{B_r(x_0)} |\nabla v|^2 \, d\mu_a\bigg)^{\frac12}.$$

Since $f \in L^s(\Omega, |x|^{-bp})$ for some $s > p/(p-2)$ we may use Hölder's inequality with conjugate exponents $s(p-1)/p$ and

$$\frac{s(p-1)}{s(p-1)-p} = \frac{p-1}{1+(p-2-\frac{p}{s})}$$

and Lemma A.1 with $\varepsilon = 2(p-2-p/s)/p$ to obtain

$$\begin{aligned} \int_{B_r(x_0)} |\nabla v|^2 \, d\mu_a &\le \mathcal{C}^2_{a,b,N}\bigg(\int_{B_r(x_0)} |x|^{-bp}|f|^s\bigg)^{\frac2s}\bigg(\int_{B_r(x_0)} |x|^{-bp}\bigg)^{\frac2p+\varepsilon} \\ &\le Cr^{-2+N\varepsilon}\max(r,|x_0|)^{-bp\varepsilon}\mu_a(B_r(x_0))\bigg(\int_{B_r(x_0)} |x|^{-bp}|f|^s\bigg)^{\frac2s}. \end{aligned} \tag{3.8}$$

From (2.7) we deduce for any $0 < \rho \le r$

$$\begin{aligned} \fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a &\le 4 \fint_{B_\rho(x_0)} |\nabla w|^2 \, d\mu_a + 4 \fint_{B_\rho(x_0)} |\nabla v|^2 \, d\mu_a \\ &\le 4C_{(2.7)}\big(\frac{\rho}{r}\big)^{2\alpha_h-2} \fint_{B_r(x_0)} |\nabla w|^2 \, d\mu_a + 4\mu_a(B_\rho(x_0))^{-1}\int_{B_r(x_0)} |\nabla v|^2 \, d\mu_a \end{aligned} \tag{3.9}$$

Since w minimizes the Dirichlet integral we may replace w in (3.9) by u. If we further estimate the integral containing v in (3.9) using (3.8) we get

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C\biggl(\mu_a(B_\rho(x_0))\mu_a(B_r(x_0))^{-1}\left(\frac{\rho}{r}\right)^{-2+2\alpha_h} \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a + r^{-2+N\varepsilon} \max(r,|x_0|)^{-bp\varepsilon} \mu_a(B_r(x_0) \|f\|^2_{L^s(B_r(x_0),|x|^{-bp})}\biggr).$$

We estimate the term $\max(r,|x_0|)^{-bp\varepsilon}$ by $r^{-bp\varepsilon}$ if $b \ge 0$ and in the case $b < 0$ by a constant $C(\Omega)$. For the rest of the proof we will consider the more interesting situation $b \ge 0$. The case $b < 0$ may be treated analogously.
Lemma A.2 with $\Phi(\rho) := \int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a$ gives for $0 < \rho < r \le r_0 := \operatorname{dist}(x_0, \partial\Omega)$

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C(\alpha)\biggl(\frac{\mu_a(B_\rho(x_0))}{\mu_a(B_r(x_0))}\left(\frac{\rho}{r}\right)^{-2+2\alpha} \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a + \rho^{-2+(N-bp)\varepsilon} \mu_a(B_\rho(x_0) \|f\|^2_{L^s(\Omega,|x|^{-bp})}\biggr).$$

We take a cut-off function $\xi \in C_c^\infty(B_r(x_0))$ such that $\xi \equiv 1$ in $B_{r/2}(x_0)$, $0 \le \xi \le 1$, $\|\nabla \xi\|_\infty \le 2r^{-1}$ and define $\varphi := \xi^2 u$. Testing with φ and using (1.3) and Hölder's inequality we get

$$\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \le \mathcal{C}_{a,b,N} \|f\|_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} \|\xi^2 u\|_{\mathcal{D}_a^{1,2}(\Omega)} + \|u \nabla \xi\|_{L^2(B_r(x_0),\, d\mu_a)} \|\nabla u \xi\|_{L^2(B_r(x_0),\, d\mu_a)}.$$

We divide by $\|\nabla u \xi\|_{L^2(B_r(x_0),\, d\mu_a)}$ and obtain

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a &\le \frac{2\mathcal{C}^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} \|\xi^2 u\|^2_{\mathcal{D}_a^{1,2}(\Omega)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + \|u \nabla \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)} \\
&\le \frac{2\mathcal{C}^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} \|\xi^2 u\|^2_{\mathcal{D}_a^{1,2}(\Omega)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + 4r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)} \\
&\le 2\mathcal{C}^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} \left(\frac{8r^{-2}\|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}}{\|\nabla u \xi\|^2_{L^2(B_r(x_0),\, d\mu_a)}} + 2\right) \\
&\quad + 4r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}.
\end{aligned}$$

Thus

$$\int_{B_r(x_0)} |\nabla u|^2 \xi^2 \, d\mu_a \le 8\mathcal{C}^2_{a,b,N} \|f\|^2_{L^{\frac{p}{p-1}}(\Omega,|x|^{-bp})} + 4r^{-2} \|u\|^2_{L^2(B_r(x_0),\, d\mu_a)}.$$

Taking $r = r_0$ we have for $0 < \rho \le r_0/2$

$$\fint_{B_\rho(x_0)} |\nabla u|^2 \, d\mu_a \le C(N,a,\Omega,r_0)\left(\int_\Omega |u|^2 \, d\mu_a + \|f\|^2_{L^{\frac{2p}{p-2}}(\Omega,|x|^{-bp}\,dx)}\right)\rho^{-2+2\alpha}$$

From the above estimate, Corollary 3.2 and, the fact that $(N - bp)2/p = N - 2 - 2a$ we derive the desired conclusion. $\square$

4. A Brezis-Kato type Lemma

As in [1] we prove the following lemma to start an iteration procedure.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open, a, b and p satisfy (1.2), and $q > 2$. Suppose $u \in D^{1,2}_a(\mathbb{R}^N) \cap L^q(\Omega, |x|^{-bp})$ is a weak solution of*

$$-\mathrm{div}\left(|x|^{-2a}\nabla u\right) - \frac{V(x)}{|x|^{bp}}\, u = \frac{f(x)}{|x|^{bp}} \quad \text{in } \Omega, \tag{4.1}$$

where $f \in L^q(\Omega, |x|^{-bp})$ and V satisfies for some $\ell > 0$

$$\int_{|V(x)|\geq \ell} |x|^{-bp}|V|^{\frac{p}{p-2}} + \int_{\Omega\setminus B_\ell(0)} |x|^{-bp}|V|^{\frac{p}{p-2}} \leq \min\left\{\frac{1}{8}\mathcal{C}^{-1}_{a,b}, \frac{2}{q+4}\mathcal{C}^{-1}_{a,b,N}\right\}^{\frac{p}{p-2}}. \tag{4.2}$$

Then for any $\Omega' \Subset \Omega$

$$\|u\|_{L^{\frac{pq}{2}}(\Omega',|x|^{-bp})} \leq C(\ell, q, \Omega')\|u\|_{L^q(\Omega,|x|^{-bp})} + \|f\|_{L^q(\Omega,|x|^{-bp})}. \tag{4.3}$$

If, moreover, $u \in \mathcal{D}^{1,2}_a(\Omega)$ then (4.3) remains true for $\Omega' = \Omega$.

Proof. Hölder's inequality, (1.3) and (4.2) give for any $w \in \mathcal{D}^{1,2}_a(\Omega)$

$$\begin{aligned}
\int_\Omega |x|^{-bp}|V(x)|w^2 &\leq \ell \int\limits_{\substack{|V(x)|\leq \ell \text{ and}\\ x\in\Omega\cap B_\ell(0)}} |x|^{-bp}w^2 + \int\limits_{\substack{|V(x)|\geq \ell \text{ or}\\ x\in\Omega\setminus B_\ell(0)}} |x|^{-b(p-2)}|V||x|^{-2b}w^2 \\
&\leq \ell \int_{\Omega\cap B_\ell(0)} |x|^{-bp}w^2 + \left(\int_\Omega \frac{w^p}{|x|^{bp}}\right)^{\frac{2}{p}} \Bigg(\int\limits_{\substack{|V(x)|\geq \ell \text{ or}\\ x\in\Omega\setminus B_\ell(0)}} |x|^{-bp}|V|^{\frac{p}{p-2}}\Bigg)^{\frac{p-2}{p}} \\
&\leq \ell \int_{\Omega\cap B_\ell(0)} |x|^{-bp}w^2 + \min\left(\frac{1}{8}, \frac{2}{q+4}\right)\int_\Omega |x|^{-2a}|\nabla w|^2.
\end{aligned} \tag{4.4}$$

Suppose now that $u \in L^q(\Omega, |x|^{-bp})$. Fix $\Omega' \Subset \Omega$ and a nonnegative cut-off function η, such that $\mathrm{supp}(\eta) \Subset \Omega$ and $\eta \equiv 1$ on Ω'. Set $u^n := \min(n, |u|) \in D^{1,2}_a(\mathbb{R}^N)$ and test (4.1) with $u(u^n)^{q-2}\eta^2 \in \mathcal{D}^{1,2}_a(\Omega)$. This leads to

$$\begin{aligned}
&(q-2)\int_\Omega |x|^{-2a}\eta^2|\nabla u^n|^2(u^n)^{q-2} + \int_\Omega |x|^{-2a}\eta^2(u^n)^{q-2}|\nabla u|^2 \\
&= \int_\Omega |x|^{-bp}V(x)\eta^2u^2(u^n)^{q-2} + \int_\Omega |x|^{-bp}f\eta^2(u^n)^{q-2}u - 2\int_\Omega |x|^{-2a}\nabla u\eta(u_n)^{q-2}\nabla\eta u.
\end{aligned}$$

We use the elementary inequality $2ab \leq 1/2a^2 + 4b^2$ and obtain

$$\begin{aligned}
&(q-2)\int_\Omega |x|^{-2a}\eta^2|\nabla u^n|^2(u^n)^{q-2} + \frac{1}{2}\int_\Omega |x|^{-2a}\eta^2(u^n)^{q-2}|\nabla u|^2 \\
&\leq \int_\Omega |x|^{-bp}V(x)\eta^2u^2(u^n)^{q-2} + \int_\Omega |x|^{-bp}f\eta^2(u^n)^{q-2}u + 4\int_\Omega |x|^{-2a}|\nabla\eta|^2u^2(u_n)^{q-2}.
\end{aligned} \tag{4.5}$$

Furthermore, an explicit calculation gives

$$\begin{aligned}
\left|\nabla\left((u^n)^{\frac{q}{2}-1}u\eta\right)\right|^2 &\leq \frac{(q+4)(q-2)}{4}(u^n)^{q-2}\eta^2|\nabla u^n|^2 + 2(u^n)^{q-2}|\nabla u|^2\eta^2 \\
&\quad + 2(u^n)^{q-2}u^2|\nabla\eta|^2 + \frac{q-2}{2}(u^n)^q|\nabla\eta|^2.
\end{aligned} \tag{4.6}$$

Let $C(q) := \min\left\{\frac{1}{4}, \frac{4}{q+4}\right\}$. From (4.5) and (4.6) we get

$$C(q)\int_\Omega \frac{\left|\nabla\left((u^n)^{\frac{q}{2}-1}u\eta\right)\right|^2}{|x|^{2a}} \le 2(2+C(q))\int_\Omega \frac{(u^n)^{q-2}u^2|\nabla\eta|^2}{|x|^{2a}} + C(q)\frac{q-2}{2}\int_\Omega \frac{(u^n)^q|\nabla\eta|^2}{|x|^{2a}}$$
$$+\int_\Omega \frac{f(x)}{|x|^{bp}}\eta^2(u^n)^{q-2}u + \int_\Omega \frac{V(x)}{|x|^{bp}}\eta^2(u^n)^{q-2}u^2. \tag{4.7}$$

Estimate (4.4) applied to $\eta(u^n)^{\frac{q}{2}-1}u$ gives

$$\int_\Omega \frac{V^+\left(\eta(u^n)^{\frac{q}{2}-1}u\right)^2}{|x|^{bp}} \le \frac{C(q)}{2}\int_\Omega \frac{\left|\nabla(\eta(u^n)^{\frac{q}{2}-1}u)\right|^2}{|x|^{2a}} + \ell\int_{\Omega\cap B_\ell(0)} \frac{(u^n)^{q-2}u^2\eta^2}{|x|^{bp}}. \tag{4.8}$$

By Hölder's inequality and convexity we arrive at

$$\int_\Omega \frac{|f|\eta}{|x|^{\frac{bp}{q}}}\frac{(u^n)^{q-2}u\eta}{|x|^{\frac{bp(q-1)}{q}}} \le \frac{q-1}{q}\int_\Omega |x|^{-bp}\eta^{\frac{q}{q-1}}|u^n|^{q\frac{q-2}{q-1}}|u|^{\frac{q}{q-1}} + \frac{1}{q}\int_\Omega |x|^{-bp}|f|^q\eta^q. \tag{4.9}$$

We use (4.8) and (4.9) to estimate the terms with f and V in (4.7), then (1.3) yields

$$\left(\int_\Omega |x|^{-bp}|u^n|^{(\frac{q}{2}-1)p}|u|^p\eta^p\right)^{\frac{2}{p}}$$
$$\le \frac{2\mathcal{C}_{a,b,N}(q-1)}{C(q)q}\int_\Omega |x|^{-bp}\eta^{\frac{q}{q-1}}|u^n|^{q\frac{q-2}{q-1}}|u|^{\frac{q}{q-1}} + \frac{2\mathcal{C}_{a,b}}{C(q)q}\int_\Omega |x|^{-bp}|f(x)|^q\eta^q$$
$$+\frac{2\ell\mathcal{C}_{a,b,N}}{C(q)}\int_{\Omega\cap B_\ell(0)} |x|^{-bp}\eta^2|u^n|^{q-2}u^2 + \frac{4\mathcal{C}_{a,b}(2+C(q))}{C(q)}\int_\Omega |x|^{-2a}|u^n|^{q-2}u^2|\nabla\eta|^2$$
$$+\mathcal{C}_{a,b,N}(q-2)\int_\Omega |x|^{-2a}|u^n|^q|\nabla\eta|^2.$$

Letting $n\to\infty$ in the above inequality (4.3) follows. Observe that if $u\in\mathcal{D}_a^{1,2}(\Omega)$ then we need not to use the cut-off η and the same analysis as above gives the estimate (4.3) for $\Omega' = \Omega$. The lemma is thereby proved. $\square$

Remark 4.2. *By Vitali's theorem V belongs to $L^{p/(p-2)}(\Omega, |x|^{-bp})$ if and only if there exists ℓ such that (4.2) is satisfied. But the constant in (4.3) depends uniformly on ℓ and not on the norm of V in $L^{p/(p-2)}(\Omega, |x|^{-bp})$.*

Proof of Theorem 1.2. We apply Lemma 4.1 with $f = 0$ and $V(x) = K(x)|u|^{p-2}$. Starting with $q = p$, the lemma gives $u \in L_{\mathrm{loc}}^{\frac{p^2}{2}}(\Omega, |x|^{-bp})$. Taking $q = \frac{p^2}{2}$, we find $u \in L_{\mathrm{loc}}^{\frac{p^3}{4}}(\Omega, |x|^{-bp})$. Iterating the process, we obtain that $u \in L_{\mathrm{loc}}^{p^{k+1}/2^k}(\Omega, |x|^{-bp})$ for any k. Let $k_0\in\mathbb{N}$ be such that $(p/2)^{k_0} \ge 2(p-1)/(p-2)$, then after k_0 steps we find that $u \in L_{\mathrm{loc}}^{\frac{2p(p-1)}{p-2}}(\Omega)$. Having this high integrability we may use Theorem 1.1 with $f(x) = K(x)|u|^{p-2}u$ to get the desired regularity of u. $\square$

Appendix A.

Lemma A.1. *Let a, b and p satisfy (1.2) and $\varepsilon > 0$. Then we have*

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \le C_{(A.1)}(N)\,\rho^{-2+\varepsilon N}(\max(\rho, |x_0|))^{-\varepsilon bp}\int_{B_\rho(x_0)} |x|^{-2a}. \tag{A.1}$$

Proof. Let us distinguish two cases.
Case 1: $\rho \geq |x_0|/2$. Since $(N-bp)(2/p+\varepsilon) = N-2-2a+\varepsilon(N-bp)$ we obtain

$$\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} \leq \left(\int_{B_{3\rho}(0)} |x|^{-bp}\right)^{2/p+\varepsilon} = C_1(N)\rho^{N-2-2a+\varepsilon(N-bp)}.$$

From the doubling property (2.1) and the fact that $B_\rho(0) \subset B_{4\rho}(x_0)$ we infer,

$$\begin{aligned}\rho^{\varepsilon(N-bp)-2}\int_{B_\rho(x_0)} |x|^{-2a} &\geq c\rho^{\varepsilon(N-bp)-2}\int_{B_{4\rho}(x_0)} |x|^{-2a}\\ &\geq c\rho^{\varepsilon(N-bp)-2}\int_{B_\rho(0)} |x|^{-2a} = C_2(N)\rho^{N-2-2a+\varepsilon(N-bp)}\end{aligned}$$

and the claim follows in Case 1.
Case 2: $\rho < |x_0|/2$. We have for all $x \in B_r(x_0)$ that $1/2 \leq |x|/|x_0| \leq 3$. Consequently,

$$\begin{aligned}\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} &\leq C_3(N)\rho^{N(2/p+\varepsilon)}|x_0|^{-2b-\varepsilon bp}\\ &\leq C_3(N)\rho^{N-2}|x_0|^{-2a}\rho^{2N/p-N+2}|x_0|^{-2(b-a)}\rho^{N\varepsilon}|x_0|^{-\varepsilon bp}\end{aligned}$$

From $r < |x_0|/2$ we get

$$\begin{aligned}\left(\int_{B_\rho(x_0)} |x|^{-bp}\right)^{2/p+\varepsilon} &\leq C_3(N)\rho^{N-2}|x_0|^{-2a}\rho^{N\varepsilon}|x_0|^{-\varepsilon bp}\\ &\leq C_4(N)\left(\int_{B_\rho(x_0)} |x|^{-2a}\right)\rho^{N\varepsilon}|x_0|^{-\varepsilon bp},\end{aligned}$$

which ends the proof. □

Lemma A.2. *Suppose Φ be a nonnegative and nondecreasing functions on $[0,R]$ such that*

$$\Phi(\rho) \leq A_1\ \mu_a\big(B_\rho(x)\big)\mu_a\big(B_r(x)\big)^{-1}\big(\frac{\rho}{r}\big)^{-\alpha}\Phi(r) + A_2\ \mu_a\big(B_r(x)\big)r^{-\beta}, \tag{A.2}$$

for any $0 < \rho \leq r \leq R$, where A_1, A_2, α and β are positive constants satisfying $\alpha < \beta$. Then for any $\gamma \in (\alpha,\beta)$ there exists a constant $C_{(A.3)} = C_{(A.3)}(A_1,\alpha,\beta,\gamma)$ independent of x and r such that for $0 < \rho \leq r \leq R$

$$\Phi(\rho) \leq C_{(A.3)}\left(\mu_a\big(B_\rho(x)\big)\mu_a\big(B_r(x)\big)^{-1}\big(\frac{\rho}{r}\big)^{-\gamma}\Phi(r) + A_2\ \mu_a\big(B_\rho(x)\big)\rho^{-\beta}\right). \tag{A.3}$$

Proof. Fix $\gamma \in (\alpha,\beta)$ and set $\tau := \min(A_1^{-1/(\gamma-\alpha)}, 1/2)$. Then we have for $0 < r \leq R$

$$\Phi(\tau r) \leq \mu_a(B_{\tau r}(x))\mu_a(B_r(x))^{-1}\tau^{-\gamma}\Phi(r) + A_2 r^{-\beta}\mu_a(B_r(x)).$$

Hence we may estimate for $k \in \mathbb{N}$

$$\begin{aligned}\Phi(\tau^{k+1}r) &\leq \mu_a(B_{\tau^{k-1}r}(x))\mu_a(B_{\tau^k r}(x))^{-1}\tau^{\beta-\gamma})\\ &\leq \mu_a(B_{\tau^{k+1}r}(x))\mu_a(B_r(x))^{-1}\tau^{-(k+1)\gamma}\Phi(r) + A_2(\tau^k r)^{-\beta}\mu_a(B_{\tau^k r}(x))\\ &\quad\cdot\sum_{j=0}^{k}\underbrace{\mu_a(B_{\tau^{k+1}r}(x))\mu_a(B_{\tau^k r}(x))^{-1}}_{\leq 1}\ \underbrace{\mu_a(B_{\tau^{k-j}r}(x))\mu_a(B_{\tau^{k-j+1}r}(x))^{-1}}_{\leq C_{(2.1)}(\tau)\text{ by (2.1)}}\tau^{(\beta-\gamma)j}\\ &\leq C_{(2.1)}(\tau)\mu_a(B_{\tau^{k+2}r}(x))\mu_a(B_r(x))^{-1}\tau^{-(k+1)\gamma}\Phi(r) + \frac{A_2 C_{(2.1)}(\tau)}{1-\tau^{\beta-\gamma}}(\tau^k r)^{-\beta}\mu_a(B_{\tau^k r}(x))\end{aligned}$$

For $0 < \rho \le r$ we may choose $k \in \mathbb{N}$ such that $\tau^{k+2} r < \rho < \tau^{k+1} r$ and obtain

$$\begin{aligned}\Phi(\rho) &\le \Phi(\tau^{k+1} r) \\ &\le C_{(2.1)}(\tau)\mu_a(B_\rho(x))\mu_a(B_r(x))^{-1}\Big(\frac{\rho}{r}\Big)^{-\gamma}\Phi(r) + \frac{A_2 C^3_{(2.1)}(\tau)}{\tau(1-\tau^{\beta-\gamma})}\mu_a(B_\rho(x))\rho^{-\beta}.\end{aligned}$$

□

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